THREE-DIMENSIONAL ISOLATED QUOTIENT SINGULARITIES IN ODD CHARACTERISTIC

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Abstract. Let a finite group G act linearly on a finite dimensional vector space V over an algebraically closed field k of characteristic $p > 2$. Assume that the quotient V/G is an isolated singularity. In the case when p does not divide the order of G, isolated singularities V/G are completely classified and their classification reduces to Zassenhaus-Vincent-Wolf classification of isolated quotient singularities over $\mathbb{C}$. In the present paper we show that if $\dim V = 3$, then also in the modular case $p \,|\, |G|$ classification of isolated quotient singularities reduces to Zassenhaus-Vincent-Wolf classification. Some remarks on modular quotient singularities in other dimensions and in even characteristic are also given.

1. Introduction

Let V be an algebraic variety defined over an algebraically closed field k of characteristic p. Let G be a finite group acting on V and $P \in V$ a fixed closed point of this action. Under these assumptions there is a well-defined quotient algebraic variety V/G. Denote by $\pi\colon V \to V/G$ the natural projection. We say that $Q = \pi(P) \in V/G$ is an *isolated singularity*, if the variety V/G is singular at Q (i. e., the local ring $\mathcal{O}_{V/G,Q}$ is not regular), and there are no other singular points of V/G in some Zariski open neighborhood of Q. If characteristic p does not divide the order $|G|$ of G, isolated quotient singularities are completely classified up to formal or, when $k = \mathbb{C}$ is the field of complex numbers, up to analytic equivalence. The classification over $\mathbb{C}$ was obtained by H. Zassenhaus, G. Vincent, and J. Wolf as a part of classification of manifolds of constant positive curvature. It is summarized in Wolf's book [12], Chapters 5, 6, 7 (see also our survey in [11], which is written from a point of view of singularity theory). It is not hard to see that Zassenhaus-Vincent-Wolf classification covers also isolated quotient singularities over arbitrary algebraically closed field k of characteristic 0 and of prime characteristic p where p does not divide $|G|$. The modular case $p \,|\, |G|$ remains open.

The first difficulty in the modular case is that the action of G on V is not in general linearizable in a formal neighborhood of a fixed point P. Still a lot of isolated quotient singularities arise as quotients by a nonlinear action of a modular group. Thus the classification problem does not reduce to a problem of linear representation theory as in the nonmodular case. Nonlinear actions are not considered in this paper, so from now on we assume that V is a vector space and G is a subgroup of the general linear group $GL(V)$. The

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second difficulty lies in the fact that the inverse Chevalley-Shephard-Todd Theorem does not hold in the modular case. A linear map $g\colon V \to V$ is called a *pseudoreflection* if it has finite order and the set of points fixed by g is a hyperplane. By $S(V^*)$ we denote the symmetric algebra of the dual space V^* of V.

Theorem 1.1 (Chevalley-Shephard-Todd Theorem, [2], Theorem 7.2.1). *Let V be a finite dimensional vector space over a field k of characteristic p, $p \geq 0$, and G a subgroup of $GL(V)$. Then if the ring of invariants $S(V^*)^G$ of G is polynomial (and in particular V/G is nonsingular), then the group G is generated by pseudoreflections. If the characteristic p does not divide the order of the group G, then the converse also holds.*

In the modular case, a quotient V/G may be singular even for G generated by pseudoreflections, as is shown by the example of the symmetric group S_6 in its 4-dimensional irreducible 2-modular representation obtained as the nontrivial constituent of the permutation module, see [8], Example 2.2.

One can check that in the example mentioned above the quotient singularity V/S_6 is not isolated. In fact, this is a general phenomenon as follows from a remarkable result of G. Kemper and G. Malle:

Theorem 1.2 ([8], Main Theorem). *Let V be a finite dimensional vector space and G a finite irreducible subgroup of $GL(V)$. Then $S(V^*)^G$ is a polynomial ring if and only if G is generated by pseudoreflections and the pointwise stabilizer in G of any nontrivial subspace of V has a polynomial ring of invariants.*

If the condition "irreducible" could be eliminated from the statement of Theorem 1.2, the classification of isolated (linear) quotient singularities would reduce to the nonmodular case, that is, to Zassenhaus-Vincent-Wolf classification. Moreover, in Section 3 we prove the following equivalence.

Theorem 1.3. *The following two statements are equivalent:*

(1) *Kemper-Malle Theorem 1.2 holds for all, not only irreducible, finite linear groups G;*
(2) *let G be a finite subgroup of $GL(V)$ and H the normal subgroup of G generated by pseudoreflections. If V/G is an isolated singularity, then V/H is nonsingular, $p(=\operatorname{char} k) \nmid |G/H|$, and*
$$V/G \simeq (V/H)/(G/H),$$
i. e., any isolated quotient singularity V/G is naturally isomorphic to a nonmodular quotient singularity.

It should be noted that the induced action of G/H on V/H may be nonlinear, but since the group G/H is nonmodular, its action can be locally formally linearized in a neighborhood of the fixed point.

In Kemper-Malle Theorem 1.2, the ground field k need not be algebraically closed. But it is easy to see that the theorem holds for general k if and only if it holds for its algebraic closure $\bar{k}$. Thus there is no loss of generality in the assumption $k = \bar{k}$.

In the sequel, we call statement (1) of Theorem 1.3 *Kemper-Malle conjecture.* Apart from irreducible groups, G. Kemper and G. Malle proved it for

groups G acting on 2-dimensional vector space V (for 2-dimensional groups in characteristic $p > 3$ it was first proved by H. Nakajima [10]), for some indecomposable groups, and showed that it suffices to prove the conjecture in general indecomposable case ([8]). We are motivated by the problem of classifying the isolated quotient singularities and find Kemper-Malle conjecture to be a key for the modular case. Indeed, as follows from Theorem 1.3, the meaning of the conjecture is that if it holds, then *essentially modular isolated (linear) quotient singularities do not exist*, that is those that exist are in fact isomorphic to nonmodular singularities. On the other hand, if statement (2) of Theorem 1.3 could be proved with methods of algebraic geometry, this would imply the conjecture. We succeeded only in dimension 3 and odd characteristic $p > 2$. Our method relies on the classification of 2-dimensional groups generated by transvections and does not generalize to higher dimensions. Our main result is the following

Theorem 1.4. *Let V be a 3-dimensional vector space over an algebraically closed field of characteristic $p > 2$. Let G be a finite subgroup of $GL(V)$ generated by pseudoreflections. Then if the quotient V/G is singular, the singularity is not isolated, i. e., Kemper-Malle conjecture holds for such groups.*

Corollary 1.5. *If $\dim V = 3$, $p > 2$, and G is a finite subgroup of $GL(V)$ such that $0 \in V/G$ is an isolated singularity, then $0 \in V/G$ is formally isomorphic to one of nonmodular isolated quotient singularities in Zassenhaus-Vincent-Wolf classification.*

In fact, our result is a bit stronger than Theorem 1.4 and is also applicable to some groups in even characteristic; see Section 4 for the precise formulation.

The paper is organized as follows. In Section 2 we collect some standard material on quotient singularities used in the consequent sections. In Section 3 we prove Theorem 1.3. We also show that Kemper-Malle conjecture holds for groups G possessing a 1-dimensional invariant subspace, in particular, for Abelian G. Section 4 is devoted to the proof of Theorem 1.4.

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2. Preliminaries

If not stated otherwise, k denotes in this section a field of characteristic $p \geq 0$, not necessarily algebraically closed.

Lemma 2.1. *Let a finite group G act linearly on the polynomial ring $R = k[x_1, \dots, x_n]$. Then the ring of invariants R^G is polynomial if and only if R^G is regular at the maximal ideal $\mathfrak{m} \cap R^G$, where $\mathfrak{m} = (x_1, \dots, x_n)$.*

Proof. Only the sufficiency needs a proof. Suppose that R^G is regular at $\mathfrak{m} \cap R^G$, but R^G is not polynomial. But R^G is a finitely generated k-algebra (see, e. g., [2], Theorem 1.3.1), so let $f_1, \dots, f_m$ be a minimal set of generators of R^G. Let $g_1(y_1, \dots, y_m), \dots, g_r(y_1, \dots, y_m)$ be a generating set of all relations between $f_1, \dots, f_m$. Note that all the polynomials f_i, $i = 1, \dots, m$, can be

chosen homogeneous of positive degree, whereas g_j, $j = 1, \dots, r$, weighted homogeneous with

$$\text{weight of } y_i = \deg f_i.$$

It follows that $g_j(0, \dots, 0) = 0$ for all $j = 1, \dots, r$. Moreover, all the monomials of g_j have degree > 1, because otherwise the set of generators $f_1, \dots, f_m$ would not be minimal. Thus by Jacobian criterion the ring

$$k[y_1, \dots, k_m] \simeq R^G$$

is not regular at 0, a contradiction. □

Now consider two algebras A and B without zero divisors over a field k, where A is a subalgebra of B. Let $\mathfrak{m} \subset A$ and $\mathfrak{n} \subset B$ be maximal ideals such that $\mathfrak{n} \cap A = \mathfrak{m}$. Denote by $j\colon A \to B$ the inclusion, by $\widehat{A}$ and $\widehat{B}$ the formal completions of A at $\mathfrak{m}$ and B at $\mathfrak{n}$ respectively, and by $\widehat{j}\colon \widehat{A} \to \widehat{B}$ the map induced by j.

Lemma 2.2. *Assume that the following conditions are satisfied:*

1. *A and B are Noetherian;*
2. *$A/\mathfrak{m} = B/\mathfrak{n} = k$;*
3. *B is unramified at $\mathfrak{n}$ over A, that is $\mathfrak{m}B_\mathfrak{n} = \mathfrak{n}B_\mathfrak{n}$, where $B_\mathfrak{n}$ is the localization of B at $\mathfrak{n}$.*

Then $\widehat{j}$ is an isomorphism.

Proof. Let

$$j_n\colon \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{n}^n/\mathfrak{n}^{n+1},\ n \geq 0,$$

be the natural map induced by j. By [1], Lemma 10.23, it is enough to show that j_n is an isomorphism for each $n \geq 0$. Note that

$$\mathfrak{n}^n/\mathfrak{n}^{n+1} \simeq (\mathfrak{n}B_\mathfrak{n})^n/(\mathfrak{n}B_\mathfrak{n})^{n+1},$$

thus surjectivity of j_n follows easily from condition (3). Let us prove injectivity of j_n.

Assume on the contrary that j_n is not injective for some $n \geq 0$. This means that there is an element $a \in \mathfrak{m}^n$ such that $a \notin \mathfrak{m}^{n+1}$, but $a \in \mathfrak{n}^{n+1}$. Considering a as an element of $B_\mathfrak{n}$ and using again condition (3), we can write a as

$$a = \sum b_i a_i,$$

where $b_i \in B_\mathfrak{n}$, $a_i \in \mathfrak{m}^{n+1}$. Now use condition (2) and the fact that the field k is contained in A and B to write $b_i = b_i^0 + b_i'$, $b_i^0 \in k$, $b_i' \in \mathfrak{n}B_\mathfrak{n}$. This allows to rewrite a as

$$a = \sum b_i^0 a_i + b',$$

where, on the one hand, b' is an element of A not contained in $\mathfrak{m}^{n+1}$ (otherwise we would have $a \in \mathfrak{m}^{n+1}$), on the other hand, b' is contained in $(\mathfrak{n}B_\mathfrak{n})^{n+2} \cap B = \mathfrak{n}^{n+2}$. Moreover, $b' \neq 0$ because otherwise a would not be contained in $\mathfrak{n}^{n+1}$. Applying the same argument to b' we get $0 \neq b'' \in \mathfrak{n}^{n+3}$, $b'' \in \mathfrak{m}^n$, $b'' \notin \mathfrak{m}^{n+1}$. It follows that $\mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$ contains nonzero elements of $\mathfrak{n}^N$ for arbitrary large N.

The ideals $\mathfrak{n}^N$ are k-vector subspaces of B. Condition (1) implies that $V = \mathfrak{m}^n/\mathfrak{m}^{n+1}$ is a finite dimensional k-vector space. Thus

$$V_N = (\mathfrak{n}^N \cap \mathfrak{m}^n)/\mathfrak{m}^{n+1},\ N > n + 1,$$

is a descending sequence of subspaces of V. This sequence stabilizes on some subspace W of V. We saw above that each member of the sequence has a nonzero element, thus $W \neq 0$. On the other hand, by Krull's theorem $\cap_{n\geq 0}\mathfrak{n}^n = (0)$, so we must have $W = 0$. This contradiction proves injectivity of $\bar{j}_n$ and hence the lemma. □

Algebraic Lemma 2.2 can be applied to quotient singularities in the following way. Let G be a finite group that acts linearly on a vector space V. Let $P \in V$ be a (closed) point and H the stabilizer of P in G. Consider the quotients $V/H = \operatorname{Spec} S(V^*)^H$ and $V/G = \operatorname{Spec} S(V^*)^G$. Let $Q \in V/H$ and $R \in V/G$ be the images of P under the natural projections $p_H\colon V \to V/H$ and $p_G\colon V \to V/G$. In general H is not normal in G, so there is no natural group action on V/H, but anyway there is a morphism $\varphi\colon V/H \to V/G$ that makes the following diagram commutative:

$$\begin{tikzcd} V \arrow[dd, "p_G"'] \arrow[dr, "p_H"] & \\ & V/H \arrow[dl, "\varphi"] \\ V/G & \end{tikzcd}$$

Let $\widehat{\mathcal{O}}_{V/H,Q}$ and $\widehat{\mathcal{O}}_{V/G,R}$ be the complete local rings of the points $Q \in V/H$ and $R \in V/G$ respectively.

Lemma 2.3. *Assume that the ground field k is infinite. Then the map*

$$\widehat{\varphi}\colon \widehat{\mathcal{O}}_{V/H,Q} \to \widehat{\mathcal{O}}_{V/G,R}$$

induced by φ is an isomorphism.

Proof. Let $\mathfrak{m}_P$ be the maximal ideal of the point $P \in V$, $\mathfrak{m}_Q = \mathfrak{m}_P \cap S(V^*)^H$, $\mathfrak{m}_R = \mathfrak{m}_P \cap S(V^*)^G$. Then apply Lemma 2.2 to the algebras $A = S(V^*)^G$, $B = S(V^*)^H$, and the ideals $\mathfrak{m}_R$ and $\mathfrak{m}_Q$. We have to check only condition (3) of Lemma 2.2, i. e., that B is unramified at $\mathfrak{m}_Q$ over A.

First let us show that the ideal $\mathfrak{m}_Q$ is generated by polynomials $f_1, \ldots, f_n \in S(V^*)$ such that for all i, $1 \leq i \leq n$, $f_i(P) = 0$, but for all $g \in G$, $g \notin H$, $f_i(gP) \neq 0$. Indeed, choose a linear function l on V such that $l(P) = 0$ but $l(gP) \neq 0$ for all $g \in G$, $g \notin H$. Such choice is possible since k is infinite. Consider an invariant

$$L(v) = \prod_{h\in H} l(hv)$$

of the group H. Suppose now that $f'_1, \ldots, f'_{n-1}$ is any generating set of the ideal $\mathfrak{m}_Q$. Then one may choose $c_1, \ldots, c_{n-1} \in k$ so that the system

$$f_1 = f'_1 + c_1 L, \ \ldots, \ f_{n-1} = f'_{n-1} + c_{n-1} L, \ f_n = L$$

of generators of $\mathfrak{m}_Q$ has the desired property.

Now for each i, $1 \leq i \leq n$, consider

$$g_i(v) = N_H^G(f_i)(v) = \prod_{g} f_i(gv),$$

where g runs over a system of representatives of all right classes of G modulo H. By construction g_i is an invariant of G and belongs to $\mathfrak{m}_R$. Since f_i is an invariant of H,

$$\frac{g_i}{f_i} = \prod_{g \notin H} f_i(gv),$$

where the product is taken over all representatives of nontrivial right classes of G modulo H, is also an invariant of H. Moreover, $g_i/f_i(Q) \neq 0$, thus it is a unit in the local ring $\mathcal{O}_{V/H,Q}$. It follows that $g_1, \ldots, g_n$ generate the ideal $\mathfrak{m}_Q \cdot \mathcal{O}_{V/H,Q}$. □

Lemma 2.4. *Let a finite group G act linearly on a finite dimensional vector space V. Suppose that W is a subspace of V which is pointwise fixed by G. Let $P_1 \in W$ and $P_2 \in W$ be two (closed) points, and $Q_1 \in V/G$, $Q_2 \in V/G$ their images under the natural projection $p\colon V \to V/G$. Then the local rings of Q_1 and Q_2 are isomorphic:*

$$\mathcal{O}_{V/G,Q_1} \simeq \mathcal{O}_{V/G,Q_2}.$$

Proof. Let v be a vector of V that joins P_1 to P_2: $P_1 + v = P_2$. The translation by vector v is an automorphism of V as a scheme. Since $v \in W$, this translation can be pushed forward along p to an automorphism of V/G that maps the point Q_1 to Q_2. The lemma follows. □

3. Kemper-Malle conjecture and isolated quotient singularities

In this section the field k will be assumed algebraically closed. Let us prove Theorem 1.3. First consider the implication (2) ⇒ (1). Let G be a subgroup of $GL(V)$ generated by pseudoreflections. By Lemma 2.1, the ring $S(V^*)^G$ is polynomial if and only if V/G is nonsingular at the image of the origin. But then V/G is nonsingular everywhere. Indeed, the singular set is closed, and V/G can be singular only at the image of a linear subspace of V. Then our theorem follows from Chevalley-Shephard-Todd Theorem 1.1 and Lemma 2.3.

If W is a subspace of V, denote by $\mathrm{Fix}(W)$ the pointwise stabilizer of W in G. In the sequel we refer to $\mathrm{Fix}(W)$ as the *fixator* of the subspace W. Suppose that $S(V^*)^{\mathrm{Fix}(W)}$ is polynomial for every nontrivial subspace W of V. Then by Lemma 2.3 V/G is nonsingular everywhere except possibly the image of the origin. But since G is generated by pseudoreflections, V/G is nonsingular by (2).

Now let us prove the implication (1) ⇒ (2). Let G be a finite subgroup of $GL(V)$ such that the singularity V/G is isolated, and H the subgroup of G generated by pseudoreflections. Then by Lemma 2.3 and Chevalley-Shephard-Todd Theorem the fixator $\mathrm{Fix}(W)$ of every nontrivial subspace of V is generated by pseudoreflections and, moreover, $V/\mathrm{Fix}(W)$ is nonsingular. Note also that $\mathrm{Fix}(W)$ is contained in H. Then, by (1), the quotient V/H is nonsingular. Further, let g be an element of G of order p^r, $r > 0$, $p = \mathrm{char}\, k$. Such an element necessarily fixes a subspace W of positive dimension. Thus $g \in \mathrm{Fix}(W)$; in particular, g is contained in H. It follows

that $p \nmid |G/H|$. The nonmodular group G/H acts naturally on V/H and

$$V/G \simeq (V/H)/(G/H).$$

Theorem 1.3 is proven.

Lemma 3.1. *Kemper-Malle conjecture holds for groups G possessing a* 1*-dimensional invariant subspace.*

Proof. In view of Theorem 1.3, we have to show that if G is generated by pseudoreflections, has a 1-dimensional invariant subspace, and V/G is singular, then the singularity V/G is nonisolated. Let W be the 1-dimensional invariant subspace of G. Note that in this case the fixator $\operatorname{Fix}(W)$ is normal in G. If $V/\operatorname{Fix}(W)$ is singular, then by Lemma 2.4 it is nonisolated. Thus by Lemma 2.3 V/G is also nonisolated.

Suppose that $V/\operatorname{Fix}(W)$ is nonsingular. If g is an element of G of order p^r, $r > 0$, then the restriction of g to the subspace W is trivial. Thus $g \in \operatorname{Fix}(W)$. It follows that the quotient group $G/\operatorname{Fix}(W)$ is nonmodular. Its action on $V/\operatorname{Fix}(W)$ can be locally formally linearized (see, e. g., [11], Lemma 2.3), and, since G is generated by pseudoreflections, the linearization of $G/\operatorname{Fix}(W)$ is also generated by pseudoreflections. Then it follows from Chevalley-Shephard-Todd Theorem that

$$V/G \simeq (V/\operatorname{Fix}(W))/(G/\operatorname{Fix}(W))$$

is nonsingular. □

Corollary 3.2. *Kemper-Malle conjecture holds for all Abelian groups generated by pseudoreflections.*

Proof. Indeed, a linear Abelian group always has a 1-dimensional invariant subspace. □

Corollary 3.3. *Let $G < GL(V)$ be a finite Abelian group such that the singularity V/G is isolated. Then V/G is formally isomorphic to a nonmodular cyclic singularity.*

Proof. We can get rid of pseudoreflections and assume that the group G is nonmodular. Then the statement follows from Zassenhaus-Vincent-Wolf classification of isolated quotient singularities, see [12] or [11]. □

Examples illustrating statement (2) of Theorem 1.3 are easy to construct.

Example 3.4. Let k be a field of characteristic 3 and $V = k^2$. Let G be a subgroup of $GL(2,k)$ generated by

$$s = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then G is isomorphic to the direct sum $\mathbb{Z}/2 \oplus \mathbb{Z}/3$, and t is a pseudoreflection (transvection, see Section 4) generating a subgroup H isomorphic to $\mathbb{Z}/3$. Basis invariants of H are

$$f_1 = x(x+y)(x+2y) \text{ and } f_2 = y,$$

where x, y is a basis of V^*. Thus V/H is nonsingular. $G/H \simeq \mathbb{Z}/2$ acts on f_1 and f_2 via $f_1 \mapsto -f_1$, $f_2 \mapsto -f_2$, thus the singularity V/G is isolated and isomorphic to a quadratic cone.

4. Proof of Theorem 1.4

Now we are going to prove our main result, Theorem 1.4. It will be a consequence of a more general Theorem 4.1. First let us recall some terminology. A pseudoreflection $g\colon V \to V$ is called a *transvection*, if g has the only eigenvalue 1. A transvection necessarily has order p equal to characteristic of the base field k. Note that if $\dim V = 2$, then any element of $GL(V)$ of order p^r, $r \geq 1$, has in fact order p and is a transvection.

Theorem 4.1. *Let V be a 3-dimensional vector space over an algebraically closed field of any characteristic p. Let G be a finite subgroup of $GL(V)$ generated by pseudoreflections. Denote by G_p the normal subgroup of G generated by all elements of order p^r, $r \geq 1$. Assume that G_p is either*

1. *irreducible on V, or*
2. *has a 1-dimensional invariant subspace U, or*
3. *has a 2-dimensional invariant subspace W and the restriction of G_p to W is generated by two noncommuting transvections (and thus is irreducible).*

Then Kemper-Malle conjecture holds for G, i. e., if V/G is singular, then the singularity is not isolated. Moreover, if G satisfies condition (3) or condition (2) plus the induced action of G_p on V/U is generated by two noncommuting transvections, then V/G is nonsingular.

Remark 4.2. To see how Theorem 1.4 follows from Theorem 4.1, suppose that p is odd. In case (3) of Theorem 4.1, denote by H the restriction of the group G_p to W. Since $\dim W = 2$, H is an irreducible group generated by transvections. It follows from general representation theory that H is defined over a finite extension of the prime subfield of k ([5], Chapter XII). Classification of such groups is known since the beginning of the twentieth century (see, e. g., [9], § 1); it implies that H is conjugate in $GL(W)$ to the group $SL(2, q)$, $q = p^n$ – the group of 2×2 matrices of determinant 1 with entries in the Galois field $\mathbb{F}_q$, – or, in characteristic $p = 3$, H may also be conjugate to the binary icosahedral group $I^* \simeq SL(2, 5)$ in a 3-modular representation. In the last case H is conjugate to the subgroup of $SL(2, 9)$ generated by two transvections

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

where $\lambda^2 = -1$. Each group $SL(2, q)$ for odd q is also generated by two appropriate noncommuting transvections. In characteristic 2, each group generated by two noncommuting transvections is conjugate to an imprimitive group generated by

$$t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } s = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix},$$

where $x \neq 0, 1$ is an element of the field $\mathbb{F}_{2^n}$. Obviously, this group is isomorphic to the dihedral group D_n. The group $SL(2, 2^n)$, $n > 1$, is not generated by two transvections, whereas over an algebraically closed field the group $SL(2, 2)$ is conjugate to an imprimitive group described above.

Proof. The case of irreducible groups G is proved by G. Kemper and G. Malle in [8]. The proof in case (2) follows from Lemma 3.1. So we concentrate on the proof of case (3).

Note that if we show that the quotient V/G_p is nonsingular, then a non-modular group G/G_p generated by pseudoreflections acts naturally on the nonsingular variety V/G_p. Such an action is locally formally linearizable, thus the quotient $V/G \simeq (V/G_p)/(G/G_p)$ is also nonsingular. Therefore it is enough to prove our theorem in the case $G = G_p$.

Denote by N the kernel of the restriction map $G \to H$, so that we have an extension

$$1 \to N \to G \to H \to 1. \tag{1}$$

The group N is Abelian and consists of the matrices of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix},$$

where $a,\, b \in k$ and the basis is chosen so that the invariant subspace W is generated by the first two vectors.

Lemma 4.3. *In the conditions of case (3) of Theorem 4.1, the group G necessarily contains transvections that restrict to (nontrivial) transvections of the group H.*

Proof. According to the classification of the irreducible groups generated by transvections, the group H is one of the following: $SL(2,q)$, q is odd; $I^* \simeq SL(2,5)$ in the 3-modular representation described in Remark 4.2; the imprimitive 2-modular group also described in Remark 4.2. In fact, our proof works also for the groups $H = SL(2,2^n)$, $n > 1$, so we shall study also this case. Let us consider these possibilities one by one. In each case we assume that G does not contain transvections with nontrivial image in H, and come to a contradiction.

Case 1: $H = SL(2,q)$, q is odd. Then H contains matrices

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Let $\tilde{t}$ and $\tilde{s}$ be some lifts of t and s to G respectively. We assume that $\tilde{t}$ and $\tilde{s}$ are not transvections. Then one can choose a basis in V so that

$$\tilde{t} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} 1 & 0 & \mu \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mu \neq 0$ is an element of k. One has

$$\tilde{t}^{-1}\tilde{s}\tilde{t} = \begin{pmatrix} 0 & -1 & \mu \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{s}^{-1}\tilde{t}\tilde{s} = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$u = \tilde{t}^{-1}\tilde{s}\tilde{t}\tilde{s}^{-1}\tilde{t}\tilde{s} = \begin{pmatrix} 1 & 0 & \mu - 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

Then

$$u^{-1}\tilde{s} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

It follows that we could assume from the beginning that $\mu \in \mathbb{F}_q$. Conjugating u with appropriate elements of G one gets elements of N with arbitrary vectors $(a, b, 1)^T$, a, $b \in \mathbb{F}_q$ in place of the third column. In particular, there is an element u_1 in N with the third column $(0, -1, 1)^T$. The product $u_1\tilde{t}$ is a transvection that restricts to $t \in H$.

Case 2: $H = I^* < SL(2, 9)$. We represent the field $\mathbb{F}_9$ as the decomposition field of the polynomial

$$x^2 + x + 2$$

over $\mathbb{F}_3$. Then we can choose the transvections generating H to be

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } s = \begin{pmatrix} 1 & 0 \\ x+2 & 1 \end{pmatrix}.$$

Denote again by $\tilde{t}$ and $\tilde{s}$ some lifts of t and s to G respectively. If $\tilde{t}$ and $\tilde{s}$ are not transvections, in a suitable basis they have matrices

$$\tilde{t} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{s} = \begin{pmatrix} 1 & 0 & \mu \\ x+2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $\mu \neq 0$ is an element of k. By a routine calculation one checks that

$$(\tilde{t}\tilde{s})^5 = \begin{pmatrix} 2 & 0 & 2x+1 \\ 0 & 2 & \mu+2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\tilde{s}\tilde{t})^5 = \begin{pmatrix} 2 & 0 & 2x+2\mu+1 \\ 0 & 2 & \mu \\ 0 & 0 & 1 \end{pmatrix},$$

$$u = (\tilde{t}\tilde{s})^5(\tilde{s}\tilde{t})^5 = \begin{pmatrix} 1 & 0 & \mu \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

It follows that

$$\tilde{u} = u\tilde{s} = \begin{pmatrix} 1 & 0 & 0 \\ x+2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in G.$$

Then one has

$$(\tilde{t}\tilde{u})^5 = \begin{pmatrix} 2 & 0 & 2x+1 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad (\tilde{u}\tilde{t})^5 = \begin{pmatrix} 2 & 0 & x+2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$u_1 = (\tilde{t}\tilde{u})^5(\tilde{u}\tilde{t})^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

Then $(u_1^{-1}u)^{-1}\tilde{s}$ is a transvection that restricts to $s \in H$.

Case 3: H is imprimitive, the characteristic of k is 2. The generators t and s of H are defined in Remark 4.2. Their lifts to G can be chosen to be

$$\tilde{t} = \begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \tilde{s} = \begin{pmatrix} x & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

respectively; here $\mu \neq 0, 1$ is an element of k. One has

$$\tilde{t}^2 = \begin{pmatrix} 1 & 0 & \mu + \mu^{-1} \\ 0 & 1 & \mu + \mu^{-1} \\ 0 & 0 & 1 \end{pmatrix}, \ (\tilde{s}\tilde{t})^2 = \begin{pmatrix} 1 & 0 & \mu^{-1} + x\mu \\ 0 & 1 & \mu + x^{-1}\mu^{-1} \\ 0 & 0 & 1 \end{pmatrix} \in N.$$

Together with $\tilde{t}^2$, $(\tilde{s}\tilde{t})^2$ the group N contains also all their conjugates and products. One easily deduces that N contains all matrices with the third column

$$(f(x)(x+1)\mu, f(x^{-1})(x^{-1}+1)\mu^{-1}, 1)^T$$

for all polynomials f with coefficients in $\mathbb{F}_2$. If the minimal polynomial g of x over $\mathbb{F}_2$ is reciprocal, that is $g(x^{-1}) = 0$, then if $h(x)(x+1) = 1$ for some polynomial h, then also $h(x^{-1})(x^{-1}+1) = 1$. It follows that N contains a matrix u with the third column $(\mu, \mu^{-1}, 1)^T$. Then $u\tilde{t}$ is a transvection that restricts to $t \in H$. If g is not reciprocal, then, taking $f = g$, we see that N contains a matrix with the third column

$$(0, g(x^{-1})(x^{-1}+1)\mu^{-1}, 1)^T.$$

Its conjugates and their combinations produce also a matrix with the third column $(0, \mu^{-1}, 1)^T$. Taking f to be the minimal polynomial for x^{-1}, we find also a matrix in N with the third column $(\mu, 0, 1)$. Then it again easily follows that G has a transvection that restricts to $\tilde{t}$.

Case 4: $H = SL(2, 2^n)$, $n > 1$. Since $SL(2, 2)$ is a subgroup of each $SL(2, 2^n)$, it is enough to prove the lemma for $H = SL(2, 2)$. But this is already proved above since $SL(2, 2)$ is conjugate to an imprimitive group. □

Lemma 4.4. *The group G contains transvections that restrict to generators of the group H. This holds also for $H = SL(2, 2^n)$, $n > 1$.*

Proof. The group $H = SL(2, 2^n)$, as well as imprimitive groups H and $H = I^* < SL(2, 9)$, have only one conjugacy class of transvections. Thus for these groups the lemma follows directly from Lemma 4.3. The groups $H = SL(2, q)$, q is odd, have two conjugacy classes of transvections. The transvections

$$t' = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \text{ and } t'' = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},$$

a, $b \in \mathbb{F}_q$, are conjugate if and only if a and b simultaneously are or are not squares in $\mathbb{F}_q$. The two noncommuting transvections t and s generating $SL(2, q)$ can be chosen in the form

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix},$$

where λ must be an element of $\mathbb{F}_q$ not belonging to a smaller field, and $\lambda^2 \neq -1$ if $q = 9$. We have already seen in the proof of Lemma 4.3 that t is a restriction of a transvection from G. Clearly λ in s can be chosen to be or not to be a square in $\mathbb{F}_q$, so that t and s are conjugate. □

Lemma 4.5. *Extension* (1) *is a semidirect product, and the group H can be embedded in G so that it acts on V via a decomposable representation with two invariant subspaces of dimensions* 1 *and* 2.

Proof. Lift two generating transvections t and s of H to transvections $\tilde{t}$ and $\tilde{s}$ of G and consider the subgroup that they generate in G. The planes fixed by $\tilde{t}$ and $\tilde{s}$ intersect at a line not contained in the invariant subspace W. This gives the desired splitting. □

Remark 4.6. At least for $H = SL(2, q)$, q is odd, Lemma 4.5 could also be proved with a help of known results on vanishing of the first and the second group cohomology of $SL(2, q)$ with coefficients in the natural module ([4], [3]), or with a help of the complete reducibility of low dimensional modules over $SL(2, q)$ ([7]). However, we would have to study in detail the structure of N as a module over H, and consider separately the cases $H = I^*$ and H is imprimitive. This is the reason why we have preferred the elementary uniform proof presented above.

From now on, we fix some splitting of (1) and consider H as a subgroup of G. Next we are going to study the quotient V/G in two steps: first take the quotient V/N and then consider the induced action of H on V/N. By [6], Theorem 3.9.2, the invariant ring of N is polynomial.

Lemma 4.7. *The induced action of H on $V/N \simeq k^3$ is linear and decomposable with invariant subspaces of dimensions* 1 *and* 2*.*

Proof. Fix a basis for V in which the elements of H are represented by block matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let x, y, z be the dual basis of V^*. Then N acts on x, y, z via transformations

$$x \mapsto x + \lambda z, \quad y \mapsto y + \mu z, \quad z \mapsto z,$$

λ, $\mu \in k$. Clearly, $f_3 = z$ is invariant. Let f_1 and f_2 be two other invariants that together with f_3 generate $S(V^*)^N$. One can choose f_1 and f_2 homogeneous and not containing the monomial z^m with a nonzero coefficient. First let us assume that f_1 and f_2 have different degrees, say, $\deg f_2 < \deg f_1$. Then f_2 must be semiinvariant under the induced action of H. Indeed, the degree of f_2 is preserved, thus for $h \in H$ the polynomial $h \cdot f_2$ expresses only through f_2 and f_3, but also h acts only on x and y, thus $h \cdot f_2$ expresses only through itself. Moreover, since H is generated by elements of order p but any element of $\mathbb{F}_q^*$ has multiplicative order coprime to p, f_2 is in fact an invariant of H. For the same reason if

$$h \cdot f_1 = \lambda f_1 + g(f_2, f_3),$$

then $\lambda = 1$ for each $h \in H$. But then the subset of V/N defined by equations $f_2 = f_3 = 0$ is pointwise fixed under the action of H. The natural projection $V \to V/N$ is H-equivariant. In turn, this implies that G has an invariant line contained in the subspace $W = \{z = 0\}$, which contradicts the condition H to be irreducible on W.

It follows that $\deg f_1 = \deg f_2$. By the same argument as above $h \cdot f_1$ and $h \cdot f_2$ are linear combinations of f_1 and f_2. On V/N the group H acts also by block matrices, thus the representation is decomposable. It should be

also possible to prove this lemma by a direct computation of the invariants of N and the induced action of H on them. $\square$

Let $V/N = V_1 \oplus V_2$ be the decomposition of V/N into a sum of invariant H-modules, $\dim V_1 = 2$, $\dim V_2 = 1$. Since the group H acting on V is generated by transvections, the action of H on V_1 and V_2 is also generated by transvections. In particular, the action on V_2 is trivial, and the invariant ring of H acting on V_1 is polynomial (see, e. g., [8], Proposition 7.1). It follows that the invariant ring of H acting on V/N is also polynomial, and hence the invariant ring of G acting on V is polynomial.

It remains to show that if G acts on V with a 1-dimensional invariant subspace U and the induced action of G on $W = V/U$ is irreducible and generated by two noncommuting transvections, then V/G is again nonsingular. This situation is in some sense dual to the one considered above. Denote H the natural image of G in $GL(W)$ and N the corresponding kernel. We again have an extension of the form (1). In an appropriate basis N is an Abelian group of the matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where a, $b \in k$. An argument, which is similar to the one given above and which we do not write here, shows that the group G contains transvections that map to the generators of H. The proof of the following lemma is easy and left to the reader.

Lemma 4.8. *Any two transvections on a* 3*-dimensional vector space have a common* 2*-dimensional invariant subspace.*

Now if $\tilde{t}$ and $\tilde{s}$ are transvections of G that map to generators of H, consider their common 2-dimensional invariant subspace. Clearly it does not contain U, thus can be identified with W. It follows that the sequence (1) again splits.

In a suitable basis x, y, z of V^*, the group N acts via transformations of the form

$$x \mapsto x + ay + bz, \quad y \mapsto y, \quad z \mapsto z.$$

It is not hard to determine basis invariants of N. They are

$$f_1 = \prod_{g \in N} g \cdot x, \quad f_2 = y, \quad f_3 = z,$$

in particular, the invariant ring $S(V^*)^N$ is polynomial. It is also clear in this case that the induced action of H on V/N is linear and decomposable. As above, it follows that $(V/N)/H$ and hence V/G is nonsingular. This finishes the proof of Theorem 4.1. $\square$

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